Explicit Solutions for Transcendental Equations*

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Abstract. A simple method to formulate an explicit expression for the roots of any analytic transcendental function is presented. The method is based on Cauchy's integral theorem and uses only basic concepts of complex integration. One convenient method for numerically evaluating the exact expression is presented. The application of both the formulation and evaluation of the exact expression is illustrated for several classical root finding problems.

Key words. analytic functions, transcendental equations, Cauchy integral theorem

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Notation.

- $\lambda$ wavelength
- $B$ Biot number
- $C_2$ second radiation constant, $1.439 \times 10^4 \mu m{^\circ}K$
- $f$ functions of $z$
- $h, R$ center and radius of a circle in the complex plane
- $S$ Stefan number
- $w$ complex function
- $z$ independent variable
- $z_0$ value of $z$ at a singularity

1. Introduction. The determination of roots of transcendental functions is a problem commonly encountered in a broad spectrum of engineering applications (e.g., Fettis (1976), Siewert and Burniston (1972), Siewert and Phelps (1978)). There exist a wide variety of numerical methods that are useful for approximating the solution to any desired degree of accuracy. From a practical standpoint, such root finding algorithms are generally straightforward to use and provide an adequate approach for determining values for the roots. Nevertheless, it is occasionally useful to have an exact mathematical solution to the problem under consideration. For example, an explicit expression for the root allows development of analytical derivatives for uncertainty analyses and sensitivity studies. In many cases analytical derivatives provide much greater insight into the problem than numerical derivatives. Explicit expressions are also useful for checking convergence of approximate root finding techniques. Haji-Sheikh and Beck (2000) described applications of the closed form expressions that they present from several analytical heat transfer problems. This paper describes a simple method of formulating exact explicit solutions for the roots of analytic tran-

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scendental functions using Cauchy’s integral theorem from complex analysis. The method is developed and several examples are described.

General descriptions of common root finding techniques are available in most textbooks in the area of numerical analysis, e.g., Burden and Faires (1985). Leathers and McCormick (1996) addressed a different methodology for obtaining explicit solutions to several transcendental problems arising in heat transfer. The general approach used was based on methods of Muskhelishvili (1953) and was developed by Burniston and Siewert (1973). This approach “depends on formulating an appropriate Riemann problem of complex variable theory and then expressing the solution of the transcendental equation in terms of a canonical solution of that problem” (Leathers and McCormick, 1996). In some cases this approach is difficult to implement and results in expressions that are cumbersome to evaluate. This paper describes a more elemental method that uses only the most basic concepts of complex integration and that can be easily applied to a large variety of problems.

2. Development. The approach presented here determines the roots of a function by locating the singularities of the reciprocal of the function. Cauchy’s theorem (Strang, 1986) states that if a function is analytic in a simple connected region containing the closed curve $C$, the path integral of the function around $C$ is zero. On the other hand, if a function, $f(z)$, contains a single singularity at $z_o$ somewhere inside $C$ but is analytic elsewhere in the region, then the singularity can be removed by multiplying $f(z)$ by $(z - z_o)$, i.e., by a pole-zero cancellation. Cauchy’s theorem implies that the path integral of the new function around $C$ must be zero:

$$\oint_C (z - z_o)f(z) \, dz = 0.$$  
(1)

Evaluating this integral yields a first-order polynomial in $z_o$ with constant coefficients. Solving the polynomial for $z_o$ yields the location of the singularity:

$$z_o = \frac{\oint_C zf(z) \, dz}{\oint_C f(z) \, dz}. $$  
(2)

Equation (2) is an explicit expression for the singularity of the function $f(z)$. A root finding problem may be recast as a singularity at the root, and (2) yields the desired root. As a practical matter for determining roots, the evaluation of (2) may or may not compete well with approximate numerical techniques for root finding. However, the method and ease of evaluation is an entirely separate issue from the derivation of an exact, explicit solution. The notable result is that (2) provides just such an explicit solution for the root. This expression, (2), can be evaluated over any closed path and with any technique, analytical or numerical, that is convenient.

One strategy for evaluation of (2) that results in a particularly straightforward calculation uses a circle in the complex plane that circumscribes the root. The closed curve $C$ may then be described as a circle in the complex plane with center $h$ and radius $R$:

$$z = h + Re^{i\theta}, \quad dz = iRe^{i\theta} \, d\theta.$$  
(3)

The values of $h$ and $R$ do not matter as long as the circle circumscribes the root. Cauchy’s principle of the argument may be used to determine the number of roots
enclosed by the path \( C \). Equation (2) becomes

\[
(4) \quad z_o = h + R \left[ \frac{\int_0^{2\pi} w(\theta)e^{2i\theta} \, d\theta}{\int_0^{2\pi} w(\theta)e^{i\theta} \, d\theta} \right],
\]

where \( w(\theta) = f(h + Re^{i\theta}) \).

The structure of (4) makes evaluation very accessible. The definition of the \( n \)th coefficient in a complex exponential Fourier series is

\[
(5) \quad A_n = \frac{1}{2\pi} \int_0^{2\pi} x(t)e^{int} \, dt,
\]

where \( x(t) \) is the function to be represented by the series. It can be seen that the term in brackets in (4) is equal to the ratio of the second Fourier series coefficient to the first for the function \( w(\theta) \). Fourier series coefficients may be calculated easily using standard complex fast Fourier transform (FFT) functions found in most mathematical software packages. Notice that, provided that \( f(z) \) is analytic at \( h \), multiplying \( f(z) \) by a factor of \( (z - h) = Re^{i\theta} \) will not change the location of the singularities of \( f(z) \). This implies that for a given singularity the term in brackets is also equal to any ratio of the \( (j+1) \) to the \( j \)th Fourier series coefficients of \( w(\theta) \), with \( j \geq 1 \).

3. Examples. For a simple illustration, (4) will be used to derive an explicit expression for the roots of the transcendental equation

\[
(6) \quad z \tan(z) = B.
\]

This equation arises from the infinite series solution to the one-dimensional transient heat conduction problem with one adiabatic and one convective boundary condition. A similar equation comes from the solution of a longitudinal vibration problem in a uniform bar with one fixed and one attached mass boundary condition. Positive roots of the equation are required. From a consideration of the functions \( \tan(z) \) and \( 1/z \) (see Figure 1), it is apparent that for the \( n \)th root, a suitable choice for the closed path \( C \) is a circle of radius \( R = \pi/4 \) centered at \( h_n = (n - 3/4)\pi \). A straightforward way to configure the equation to provide a singularity is to simply use the inverse of a rearranged equation (6):

\[
(7) \quad f(z) = \frac{1}{z \sin(z) - B \cos(z)}.
\]

Any other convenient configuration (e.g., \( 1/[1-e^{(x \tan(x) - B)]} \)) could be used to create the singularity required for (1). Applying (4) results in an explicit expression for the \( n \)th root. This expression can be evaluated simply by calculating the FFT as described earlier, or via numerical integration. The first root for \( B = 2 \), which is \( 1.07687 \ldots \), can be evaluated correct to 11 decimal places with an FFT using 32 points. Figure 2 illustrates the effect of the number of points used in the FFT on the accuracy of the result. Each point requires one function evaluation. The number of function evaluations required for a secant method root finding algorithm using (6) is included simply as a point of interest. Clearly the secant method algorithm surpasses the FFT in computational efficiency in generating an approximation to the exact
**Fig. 1** Roots of $z \tan(z) = B$.

**Fig. 2** Convergence of FFT method of evaluating exact formulation of roots.
answer; however, (2) and (4) provide the exact answer regardless of the method used to evaluate it.

Function derivatives are commonly used in uncertainty calculations and other analytical applications. The calculation of an explicit exact derivative, \( \frac{dz_o}{dB} \), is straightforward from (4) and (7):

\[
\frac{dz_o}{d B} = R \left[ \frac{F_1(w)F_2'(w) - F_2(w)F_1'(w)}{F_1(w)^2} \right],
\]

where

\[
F_n(x) = \int_0^{2\pi} x e^{in\theta} d\theta,
\]

\[
w = f(z),
\]

\[w' = \frac{dw}{d B} = -\frac{\cos(z)}{(z \sin(z) - B \cos(z))^2}.
\]

For the first root with \( B = 2 \), the value of the derivative is \( \frac{dz_o}{d B} = 0.1504 \ldots \). Figure 3 illustrates the error associated with three different evaluations of the derivative using an FFT. Using 32 points in the FFT results in an error of less than \( 10^{-11} \). For comparison, Figure 3 includes the error in the approximate derivative calculated numerically from a secant method root finding algorithm. The numerical derivative was approximated by

\[
\frac{dz_o}{d B} \approx \frac{SR(B + \varepsilon) - SR(B - \varepsilon)}{2\varepsilon},
\]

where \( SR(B) \) represents the secant method root finding algorithm. The error is a function of the perturbation size, \( \varepsilon \). Below \( \varepsilon \approx 10^{-5} \) the round-off error in the numerical approximation impeded further improvement in the approximation.

A second illustrative example comes from Wien’s law. The wavelength of maximum emissive power for black-body radiation can be obtained from \((5 - z)e^z = 5\), where \( z = C_2/\lambda T \). While this equation needs to be solved only once, the solution provides a nice illustration of the technique of this paper.

The function \( 5 / (5 - z) \) intersects \( e^z \) at two locations. One is the trivial solution \( z = 0 \), and the other intersection must take place to the left of \( z = 5 \), since \( 5 / (5 - z) \) approaches infinity as \( z \) approaches 5. Selecting \( f(z) = 1 / ((5 - z)e^z - 5) \), the required pole-zero cancellations are achieved by multiplying by \( (z - 0)(z - z_o) \). As a result, the integrand of (1) becomes \( z(z - z_o)f(z) \). Using \( R = 5 \) and \( h = 0 \), integrating and solving for \( z_o \) leads to modifying (4) to be

\[
z_o = R \left[ \frac{\int_0^{2\pi} w(\theta)e^{i\theta} d\theta}{\int_0^{2\pi} w(\theta)e^{2i\theta} d\theta} \right],
\]

where

\[
w(\theta) = f(h + Re^{i\theta}) = \frac{1}{[(5 - (h + Re^{i\theta}))e^{(h + Re^{i\theta})} - 5]}.
\]
The term in brackets is equal to the ratio of the third to the second Fourier series coefficients of \( w(\theta) \). As before, this term is also equal to the ratio of the \((j+1)\) to the \(j\)th Fourier series coefficients of \( w(\theta) \), with \( j \geq 2 \). Evaluation of this expression with \( R = 5 \) yields \( z_0 = 4.9651 \ldots \).

A third illustration treats the equation

\[
ze^{z^2} \text{erf}(z) = S/\sqrt{\pi}.
\]

For solidification of a pure liquid, initially at its phase change temperature in a semi-infinite region, the temperature profile in the liquid, the temperature profile in the solid, and the location of the interface need to be determined as functions of time. Roots of the above transcendental equation must be determined for the solution of this problem.

Since the left side of the equation is a parabolic-shaped, even function, there will be two solutions of opposite sign but equal magnitude. This makes it possible to focus on solutions on the positive real axis without losing generality. On the real axis, \( \text{erf}(z) \approx 1 \) for \( z \geq 2 \), and the left side of the equation is approximately 100 when \( z = 2 \). Therefore, one can choose a circle with \( h = 1 \) and \( R = 1 \) for \( S \leq 100\sqrt{\pi} \). For larger \( S \), a rough approximation to the solution can be obtained by letting \( \text{erf}(z) \approx 1 \), neglecting \( \ln(z) \) when solving for \( z \) to obtain \( z \approx \sqrt{\ln(S/\sqrt{\pi})} \). Thus, \( h \) can be set to \( h = \sqrt{\ln(S/\sqrt{\pi})} \) and \( R \) is set to \( R = 0.2 \) to account for errors in the approximation. As before, \( f(z) \) is just the reciprocal of the equation.
under consideration, \( f(z) = 1/(ze^{z^2} \text{erf}(z)\sqrt{\pi} - S) \), making

\[
w(\theta) = \frac{1}{(h + Re^{i\theta})e^{(h + Re^{i\theta})^2} \text{erf}(h + Re^{i\theta})\sqrt{\pi} - S},
\]

and the desired solution is given by (4).

**Conclusion.** A simple procedure for determining explicit solutions to a number of transcendental problems has been presented. Numerical evaluation of solutions is readily accessible, requiring only a complex Fourier transform. While the computational efficiency of this procedure would not be expected to rival that of traditional approximate root finding techniques, the procedure is conceptually simple, provides an exact explicit expression for the roots, and can easily be implemented with the tools commonly available in commercial mathematical packages.

**REFERENCES**


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